Exercise 27

Plot the gradient vector field of f together with a contour map of f. Explain how they are related to each other.

$$f(x, y) = \ln(1 + x^2 + 2y^2)$$

Solution

Calculate the gradient and call it $\mathbf{F}$.

$$\begin{aligned}
\mathbf{F} &= \nabla f \\
&= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle f \\
&= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \\
&= \left\langle \frac{\partial}{\partial x}\left[\ln(1 + x^2 + 2y^2)\right], \frac{\partial}{\partial y}\left[\ln(1 + x^2 + 2y^2)\right] \right\rangle \\
&= \left\langle \frac{1}{1 + x^2 + 2y^2} \cdot \frac{\partial}{\partial x}(1 + x^2 + 2y^2), \frac{1}{1 + x^2 + 2y^2} \cdot \frac{\partial}{\partial y}(1 + x^2 + 2y^2) \right\rangle \\
&= \left\langle \frac{1}{1 + x^2 + 2y^2} \cdot (2x), \frac{1}{1 + x^2 + 2y^2} \cdot (4y) \right\rangle \\
&= \left\langle \frac{2x}{1 + x^2 + 2y^2}, \frac{4y}{1 + x^2 + 2y^2} \right\rangle
\end{aligned}$$

The vector field of this gradient is superimposed on a contour plot of $f(x, y)$. Notice that the vectors are perpendicular to each of the contours, pointing in the direction of greatest increase.